Quantum Mechanics of a Non Commutative Space

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by

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Abstract

A non-commutative space can be understood as one in which the position operators do not commute with each other. Our project considers a modified position operator, that yields a quantum theory with a minimal length uncertainty principle. This modification, which essentially represents a spacetime "fuzzyness" could be used to model phenomena introduced by sting theory and theories of quantum gravity. The minimal length uncertainty will modify the Hamiltonian, which will in turn modify the energy eigenvalues. As the minimal length scale is tiny, we use perturbation theory, which provides a powerful shortcut to calculate the energy eigenvalues for a number of interesting potentials. Finally, we apply our approach to the hydrogen atom energy splittings to place a bound on our minimal length uncertainty parameter.

Chapter 1

Introduction

1.1 Background and Motivation

The usual one dimensional quantum mechanical Heisenberg uncertainty relation is given as follows

$$\Delta x \Delta p \geq 1/2. \tag{1.1}$$

where I have chosen units such that $\hbar$ =1. This allows for the possibility of knowing Δx very well by taking the limit $\Delta p \to \infty$. We can similarly know Δp very well. It is possible, however, to modify this uncertainty relation by introducing a term proportional to $(\Delta p)^2$

$$\Delta x \Delta p \geq \frac{1}{2}(1 + a^2(\Delta p)^2), \tag{1.2}$$

which can be re-expressed as

$$\Delta x \geq \frac{1}{2}(\frac{1}{\Delta p} + a^2 \Delta p). \tag{1.3}$$

In the limit $\Delta p \to 0$ or $\Delta p \to \infty$, it follows that $\Delta x \to \infty$; it is apparent that there will be a minimum in position uncertainty given by

$$\Delta x_{min} = a. \tag{1.4}$$

Our motivation for modifying quantum mechanics in this way is to provide a low-energy description of quantum gravity and string theory [1-5]. In string theory, for example, it is assumed that strings have a finite size. Because of this finite size, it is impossible to use strings to probe distances smaller than the string scale,

$\sim a$. This inability to probe small distances suggests the existance of a minimal length uncertainty of the form Eq. (1.3). Hence, quantum mechanics with a minimal length uncertainty might be useful low-energy description. Our next step is to find new position and momentum operators that are consistent with the existence of a minimum uncertainty in position. In general, for two non-commuting operators $\hat{A}, \hat{B}$ [6]:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle i[\hat{A}, \hat{B}]\rangle^2. \tag{1.5}$$

For normal quantum mechanics, where $\langle\ \rangle$ represents an expectation value,

$$\Delta x \Delta p \geq \frac{1}{2}\langle i \times -i\rangle = \frac{1}{2}; \tag{1.6}$$

for the existence of a minimum length uncertainty

$$\Delta x_{new} \Delta p_{new} \geq \frac{1}{2}\langle i[\hat{x}_{new}, \hat{p}_{new}]\rangle, \tag{1.7}$$

where,

$$\langle i[\hat{x}_{new}, \hat{p}_{new}]\rangle = 1 + a^2 \Delta p_{new}^2. \tag{1.8}$$

Δp may be re-expressed as

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}. \tag{1.9}$$

As an example, let us make the simplifying assumption that the expectation value of the momentum is zero [1]. We see that

$$\langle i[x_{new}, p_{new}]\rangle = 1 + a^2\langle p_{new}^2\rangle. \tag{1.10}$$

This yields

$$\Delta x_{new} \Delta p_{new} \geq \frac{1}{2}(1 + a^2\langle p_{new}^2\rangle). \tag{1.11}$$

In the limit that $a^2 \to 0$, we retain the result for normal quantum mechanics. There are a number of ways of satisfying the commutation relation

$$[\hat{x}, \hat{p}] = i(1 + a^2\langle p_{new}^2\rangle). \tag{1.12}$$

[1] We do this only for presentation; without this assumption we get a slightly more complicated form for Eq. (1.3), which still is a theory with minimal length uncertainty.

where I have dropped the subscripts. One group [7] used the following parameterization

$$\begin{aligned} \hat{x}|\Psi> &= \hat{x}_0 + a^2\hat{p}_0^2\hat{x}_0|\Psi> \\ \hat{p}|\Psi> &= \hat{p}_0|\Psi> . \end{aligned} \tag{1.13}$$

Where the subscript 0 indicates the usual quantum mechanical operators, while $\hat{x}$ and $\hat{p}$ now refer to modified operators. Another group [8] used a more general form

$$\begin{aligned} \hat{x}|\Psi> &= \hat{x}_0 + a^2\hat{p}_0^2\hat{x}_0 + i\gamma\hat{p}_0|\Psi> \\ \hat{p}|\Psi> &= \hat{p}_0|\Psi>, \end{aligned} \tag{1.14}$$

where γ is included to preserve hermiticity. Our choice is one that is both simple and hermitian, and corresponds to $\gamma = a^2$

$$\begin{aligned} \hat{x}|\Psi> &= \hat{x}_0 + a^2\hat{p}_0\hat{x}_0\hat{p}_0|\Psi> \\ \hat{p}|\Psi> &= \hat{p}_0|\Psi> . \end{aligned} \tag{1.15}$$

It can be shown that

$$\langle i[\hat{x}_0 + a^2\hat{p}_0\hat{x}_0\hat{p}_0,\, \hat{p}_0]\rangle = 1 + a^2\langle p_0^2\rangle, \tag{1.16}$$

which obviously reduces to the usual quantum mechanical result for $a^2 \to 0$. This parameterization has been generalized to three dimensions [8,9]:

$$\begin{aligned} \hat{x}^m|\Psi> &= x_0^m + a^2p_0^2x_0^m + a'^2p_0^mp_0\cdot x_0 + i\gamma p_0|\Psi> \\ \hat{p}^m|\Psi> &= p_0^m|\Psi>, \end{aligned} \tag{1.17}$$

where m is an integer, 1..3. Our choice is again simple and hermitian

$$\begin{aligned} \hat{x}^m|\Psi> &= x_0^m + a^2p_0^nx_0^mp_0^n|\Psi> \\ \hat{p}^m|\Psi> &= p_0^m|\Psi>, \end{aligned} \tag{1.18}$$

where repeated indices are summed over according to the Einstein summation convention. If we set $\gamma = a^2$ and $a'^2 = 0$ in Eq. (1.17), our parameterization Eq. (1.18), follows. Previous attempts to compute energy deviations due to the introduction of

a natural length were made by solving a very complicated Schrodinger equation in momentum space [9]. These authors chose to work in momentum space because a physical interpretation of position space wave functions is not possible in these theories. It is possible to superpose several position space wave functions to make a delta function, which is in clear violation of the minimal uncertainty relation, Eq. (1.3). In some theories of this type [10], it is not even possible to find a differential operator in position space,

$$\hat{x}_{new} = f(\frac{\partial}{\partial x_{new}}) \tag{1.19}$$

that satisfies the minimal length commutation relations. What we propose to do is to express $\hat{x}$ as a function of $\hat{x}_0$ and $\hat{p}_0 = \frac{1}{i}\frac{\partial}{\partial x_0}$ which have the usual commutation relations and uncertainty relation, and treat the $\mathcal{O}(a)$ terms in $\hat{x}$ as perturbations. We find the shifted energies in various physical systems and attempt to place a bound on our small parameter, a.

Chapter 2

Theoretical Models

2.1 One Dimensional Harmonic Oscillator

Perturbation theory provides a powerful shortcut from which we can calculate the shift in energy from the shift in the Hamiltonian

$$\Delta E_n =< n \,|\Delta H|\, n > . \tag{2.1}$$

Here, $\langle n|$ is an eigenstate for the harmonic oscillator and ΔH is the change in the hamiltonian due to our new position operator. The Hamiltonian for the harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2.2}$$

where $\hat{x} = \hat{x}_0 + a^2\hat{p}_0\hat{x}_0\hat{p}_0$, and ω is the angular frequency. A trivial calculation yields

$$\Delta H = \frac{1}{2}m\omega^2a^2[xpxp + pxpx + \mathcal{O}(a^4)], \tag{2.3}$$

where x and p are understood to be the usual quantum mechanical operators and I have dropped the subscript 0. For convenience, we shall utilize the harmonic oscillator raising and lowering operators

$$x = \frac{1}{\sqrt{2m\omega}}(A + A^\dagger) \tag{2.4}$$

$$p = -i\sqrt{\frac{m\omega}{2}}(A - A^\dagger), \tag{2.5}$$

where

$$A|n>=\sqrt{n}|n-1> \tag{2.6}$$

$$A^\dagger|n>=\sqrt{n+1}|n+1>\,. \tag{2.7}$$

Ignoring terms $\mathcal{O}(a^4)$ we shall proceed in our investigation by considering each term separately

$$xpxp=-\frac{1}{4}(A+A^\dagger)(A-A^\dagger)(A+A^\dagger)(A-A^\dagger) \tag{2.8}$$

$$xpxp=-\frac{1}{4}(AA-AA^\dagger+A^\dagger A-A^\dagger A^\dagger)(AA-AA^\dagger+A^\dagger A-A^\dagger A^\dagger). \tag{2.9}$$

Because we are eventually evaluating $\Delta E_n =< n|\Delta H|n >$, we are only interested in terms with an equal number of A and $A^\dagger$. Hence, the relevant terms are

$$-\frac{1}{4}(-AAA^\dagger A^\dagger+AA^\dagger AA^\dagger-AA^\dagger A^\dagger A-A^\dagger AAA^\dagger+A^\dagger AA^\dagger A-A^\dagger A^\dagger AA). \tag{2.10}$$

Similarly,

$$pxpx=-\frac{1}{4}(A-A^\dagger)(A+A^\dagger)(A-A^\dagger)(A+A^\dagger) \tag{2.11}$$

$$pxpx=-\frac{1}{4}(AA+AA^\dagger-A^\dagger A-A^\dagger A^\dagger)(AA+AA^\dagger-A^\dagger A-A^\dagger A^\dagger) \tag{2.12}$$

And the relevant terms are

$$-\frac{1}{4}(-AAA^\dagger A^\dagger+AA^\dagger AA^\dagger-AA^\dagger A^\dagger A-A^\dagger AAA^\dagger+A^\dagger AA^\dagger A-A^\dagger A^\dagger AA). \tag{2.13}$$

Combining these results, we see that

$$\Delta H=\frac{1}{2}m\omega^2a^2\frac{1}{4}(2AAA^\dagger A^\dagger+2A^\dagger A^\dagger AA-2AA^\dagger AA^\dagger+2AA^\dagger A^\dagger A+2A^\dagger AAA^\dagger-2A^\dagger AA^\dagger A). \tag{2.14}$$

It follows that

$$\Delta E_n=\frac{1}{4}m\omega^2a^2(n(n-1)+(n+1)(n+2)-(n+1)^2+(n+1)n+n(n+1)-n^2) \tag{2.15}$$

$$\Delta E_n=\frac{1}{4}m\omega^2a^2((n+1)(n+2-n-1+2n)+n^2-n-n^2) \tag{2.16}$$

$$\Delta E_n=\frac{1}{4}m\omega^2a^2((n+1)(2n+1)-n) \tag{2.17}$$

$$\Delta E_n = \frac{1}{4} m\omega^2 a^2 (2n^2 + 2n + 1). \tag{2.18}$$

This is in agreement with results found through more complicated, non-perturbative methods in momentum space [7], [8]. Their results are as follows,

$$E_n = \omega \left[(n + \frac{1}{2}) \sqrt{1 + \frac{a^4 m^2 \omega^2}{4}} + \frac{m\omega^2 a^2}{2} (n^2 + n + \frac{1}{2}) \right]. \tag{2.19}$$

In our limit, where we ignore terms $\mathcal{O}(a^4)$,

$$E_n = \omega \left[(n + \frac{1}{2}) + \frac{m\omega^2 a^2}{2} (n^2 + n + \frac{1}{2}) \right]. \tag{2.20}$$

Clearly, then

$$\Delta E_n = \frac{1}{4} m\omega^2 a^2 (2n^2 + 2n + 1), \tag{2.21}$$

which is in agreement with Eq. (2.18). Later in this paper we shall use this result to find a bound on the natural length parameter, a.

2.2 Three Dimensional Harmonic Oscillator

The method for approximating the energy shift in the three dimensional harmonic oscillator is similar to that of the one dimensional case; however, since the energy levels for the three dimensional case are degenerate, we shall have to use degenerate perturbation theory. We shall also have to change a few of our definitions accordingly:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r_{new}^2 \tag{2.22}$$

$$x_{new}^m = x^m + a^2 p^n x^m p^n \tag{2.23}$$

$$x^m = \frac{1}{\sqrt{2m\omega}} (A_m + A_m^\dagger) \tag{2.24}$$

$$p^n = -i \sqrt{\frac{m\omega}{2}} (A_n - A_n^\dagger) \tag{2.25}$$

$$A_m |n_m\rangle = \sqrt{n_m} |n_m - 1\rangle \tag{2.26}$$

$$A_m^\dagger |n_m\rangle = \sqrt{n_m + 1} |n_m + 1\rangle \tag{2.27}$$

Finally, we define the degenerate perturbation theory matrix elements:

$$W_{ij} = \langle \Psi_i^0 | \Delta H | \Psi_j^0 \rangle \tag{2.28}$$

From this, it follows that

$$\Delta H = \frac{1}{2} m\omega^2 a^2 \left[x^m p^n x^m p^n + p^n x^m p^n x^m + \mathcal{O}(a^4) \right] \tag{2.29}$$

$$\begin{aligned} \Delta H = \frac{1}{2} m\omega^2 a^2 [& x(p_x + p_y + p_z) x(p_x + p_y + p_z) + ... \\ & + (p_x + p_y + p_z) x(p_x + p_y + p_z) x + ...] \end{aligned} \tag{2.30}$$

Where I have omitted terms $\mathcal{O}(a^4)$ and used "..." to signify similar terms involving y and z (*i.e.*, $m = 2, 3$).

$$\begin{aligned} \Delta H = & \\ -\frac{1}{8} m\omega^2 a^2 [& (A + A^\dagger)(A - A^\dagger + B - B^\dagger + C - C^\dagger) \cdot \\ & (A + A^\dagger)(A - A^\dagger + B - B^\dagger + C - C^\dagger) + ... + \\ & (A - A^\dagger + B - B^\dagger + C - C^\dagger)(A + A^\dagger) \cdot \\ & (A - A^\dagger + B - B^\dagger + C - C^\dagger)(A + A^\dagger) + ...] \end{aligned} \tag{2.31}$$

This can be reduced to the following:

$$\begin{aligned} -\frac{1}{4} m\omega^2 a^2 [& A^4 - A^2 A^{\dagger 2} - A A^\dagger A A^\dagger - A A^{\dagger 2} A - A^\dagger A A^\dagger A - A^{\dagger 2} A^2 + A^{\dagger 4} + ... \\ & + (A^2 + A A^\dagger + A^\dagger A + A^{\dagger 2})(B^2 - B B^\dagger - B^\dagger B + B^{\dagger 2} + C^2 - C C^\dagger - C^\dagger C + C^{\dagger 2}) + ...], \end{aligned} \tag{2.32}$$

where A, B, C correspond to A_1, A_2, A_3. We are now prepared to compute the matrix Eq. (2.28). For the ground state, n = 0, our matrix is one dimensional, with $\Psi^0_{j=1} = \langle 000|$ and $\Psi^0_{i=1} = |000\rangle$.

$$\Delta E = \langle 000 | \Delta H | 000 \rangle \tag{2.33}$$

$$\Delta E = \frac{9}{4} m\omega^2 a^2 \tag{2.34}$$

For the ground state, n = 0, there is only one possible energy.

For the first excited state, n = 1, there are three possible combinations of $n = n_x + n_y + n_z = 1$. This will create the following 3x3 matrix:

$$W_2 = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} \tag{2.35}$$

The basis of the matrix W_1 is $|001\rangle$, $|010\rangle$, $|100\rangle$. It can be shown that:

$$W_1 = \frac{1}{4} m\omega^2 a^2 \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} \tag{2.36}$$

Thus, for n = 1, there will be 3 states, all with energy shift:

$$\Delta E = \frac{21}{4} m\omega^2 a^2 \tag{2.37}$$

Finally, for the n = 2 case, there are 6 possible combinations of $n = n_x + n_y + n_z = 2$ This will create the following 6x6 matrix:

$$W_2 = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} & W_{26} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} & W_{36} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} & W_{46} \\ W_{51} & W_{52} & W_{53} & W_{54} & W_{55} & W_{56} \\ W_{61} & W_{62} & W_{63} & W_{64} & W_{65} & W_{66} \end{bmatrix} \tag{2.38}$$

The basis of the matrix W_2 is $|011\rangle$, $|101\rangle$, $|110\rangle$, $|002\rangle$, $|020\rangle$, $|200\rangle$. It can be shown that:

$$W_2 = \frac{1}{4} m\omega^2 a^2 \begin{bmatrix} 41 & 0 & 0 & 0 & 0 & 0 \\ 0 & 41 & 0 & 0 & 0 & 0 \\ 0 & 0 & 41 & 0 & 0 & 0 \\ 0 & 0 & 0 & 37 & -4 & -4 \\ 0 & 0 & 0 & -4 & 37 & -4 \\ 0 & 0 & 0 & -4 & -4 & 37 \end{bmatrix} \tag{2.39}$$

For the n = 2 case, there will be 6 possible states, 5 with energy

$$\Delta E = \frac{41}{4} m\omega^2 a^2 \tag{2.40}$$

and 1 with energy:

$$\Delta E = \frac{29}{4} m\omega^2 a^2 \tag{2.41}$$

The n = 0, 1, 2 states are in perfect agreement with the literature [8]:

$$\Delta E_{nl} = \frac{1}{4} m\omega^2 a^2 [2n^2 + 6n + 9 + 2l(l+1)] \tag{2.42}$$

Where we have set $\gamma = a^2$ and $a'^2 = 0$ in Eq. (1.17) and $n = 2n_r + l$ where both n and l are integers ≥ 0. Hence, to compare our result for the $n = 2$ case, we must set $n_r = 0$ and $l = 2$, yielding Eq. (2.40), or $n_r = 2$ and $l = 0$, yielding Eq. (2.41).

2.3 The Ground State of the Hydrogen Atom

The somewhat more complicated case of the hydrogen atom can be treated by considering the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{r}. \tag{2.43}$$

In previous attempts [9], the Schrodinger Equation with this Hamiltonian was re-expressed by multiplying through by r and then solving it in momentum space. This is not an option for us because of the way perturbation theory is derived. Eq. (2.43), however, can be re-expressed as

$$H = \frac{p^2}{2m} - \frac{e^2}{r_{old}} + \left(\frac{e^2}{r_{old}} - \frac{e^2}{r_{new}} \right). \tag{2.44}$$

Our perturbed hamiltonian is

$$\Delta H = \frac{e^2}{r_{old}} - \frac{e^2}{r_{new}}. \tag{2.45}$$

Similarly,

$$\Delta E = < \Psi | \frac{e^2}{r_{old}} - \frac{e^2}{r_{new}} | \Psi > . \tag{2.46}$$

It is apparent that

$$\frac{1}{r_{new}} = \frac{1}{\sqrt{x^i_{new} x^i_{new}}} = \frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)(x^i + a^2 p^j x^i p^j)}}, \tag{2.47}$$

which may be interpretted as the square root of the inverse of a Hilbert space operator, which we assume is well defined. The second term in the energy shift is then given by

$$e^2 < \Psi|\frac{1}{r_{new}}|\Psi >= e^2 < \Psi| \left[\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}\right] \left[\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}\right] |\Psi > \quad (2.48)$$

$$= e^2 \left[\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi >\right]^\dagger \left[\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}\right] |\Psi > . \quad (2.49)$$

We can now treat these two identical terms separately and recombine later to find the shifted energy.

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi >= \left[x^i - a^2(\frac{\partial}{\partial x^j} x^i \frac{\partial}{\partial x^j})\right]^{-\frac{1}{2}} |\Psi > \quad (2.50)$$

$$= \left[x^i - a^2(x^i \bigtriangledown^2 + \frac{\partial}{\partial x^i})\right]^{-\frac{1}{2}} |\Psi > . \quad (2.51)$$

The ground state wave function for the hydrogen atom is given as follows:

$$\Psi_{100} = \frac{2}{\sqrt{4\pi}} \left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-\frac{r}{a_0}} \quad (2.52)$$

We shall make use of the following simplifications

$$\bigtriangledown^2 e^{-\frac{r}{a_0}} = \frac{1}{a_0} \left(\frac{1}{a_0} - \frac{2}{r}\right) e^{-\frac{r}{a_0}}, \quad (2.53)$$

$$\frac{\partial}{\partial x^i} e^{-\frac{r}{a_0}} = -\frac{x^i}{a_0 r} e^{-\frac{r}{a_0}}, \quad (2.54)$$

where a_0 is the Bohr radius. Using these results, we see that

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi >= \left[x^i \left(1 - \frac{a^2}{a_0}\left(\frac{1}{a_0} - \frac{3}{r}\right)\right)\right]^{-\frac{1}{2}} |\Psi > . \quad (2.55)$$

We are now prepared to calculate the energy shift

$$e^2 < \Psi|\frac{1}{r_{new}}|\Psi > \quad (2.56)$$

$$= \frac{4e^2 \cdot 4\pi}{4\pi a_0^3} \int_0^\infty r^2 dr e^{-\frac{2r}{a_0}} \frac{1}{\sqrt{x^{i2}}} \left[1 - \frac{a^2}{a_0}\left(\frac{1}{a_0} - \frac{3}{r}\right)\right]^{-1}. \tag{2.57}$$

After recognizing $\sqrt{x^{i2}} = r$, we can make a change of variable to $\tilde{r} \equiv \frac{r}{a_0}$

$$e^2 < \Psi|\frac{1}{r_{new}}|\Psi >= \frac{4e^2}{a_0} \int_0^\infty \tilde{r} d\tilde{r} e^{-2\tilde{r}} \frac{1}{1 - \beta\left(1 - \frac{3}{\tilde{r}}\right)} \tag{2.58}$$

This integral can be simplified through a simple expansion,

$$\frac{1}{1 - \beta(1 - \frac{3}{\tilde{r}})} \simeq 1 + \beta(1 - \frac{3}{\tilde{r}})$$

provided we recognize that the term $\beta\left(1 - \frac{3}{\tilde{r}}\right)$ is small. This expansion is valid provided that the final result is analytic in β. We have verified the validity of this step by checking our result both analytically and numerically.

$$e^2 < \Psi|\frac{1}{r_{new}}|\Psi >= \frac{4e^2}{a_0} \int_0^\infty \tilde{r} d\tilde{r} e^{-2\tilde{r}} \left[1 + \beta\left(1 - \frac{3}{\tilde{r}}\right)\right] \tag{2.59}$$

where I have defined $\beta \equiv \frac{a^2}{a_0^2}$. The total energy shift Eq. (2.46) is then given by

$$\Delta E = \frac{4e^2}{a_0} \int_0^\infty \tilde{r} d\tilde{r} e^{-2\tilde{r}} \left[1 - \left(1 + \beta\left(1 - \frac{3}{\tilde{r}}\right)\right)\right]. \tag{2.60}$$

$$\Delta E = -\frac{4e^2\beta}{a_0} \int_0^\infty d\tilde{r} e^{-2\tilde{r}} \left(\tilde{r} - 3\right). \tag{2.61}$$

It's easy enough to see that if we set a = 0, we retain absolutely no shift in the energy spectrum. This integral is simple to compute:

$$\Delta E = -\frac{4e^2\beta}{a_0}\left(-\frac{5}{4}\right). \tag{2.62}$$

$$\Delta E_{100} = \frac{5e^2a^2}{a_0^3} \tag{2.63}$$

This result is in disagreement with the literature [9].

2.4 The First Excited State of the Hydrogen Atom

The 1S-2S energy splitting of the hydrogen atom is very well measured and can be used to place a bound on the small parameter, a. The $n = 2$ states the hydrogen atom are significantly more difficult to compute, and once again, we must use degenerate perturbation theory; however, the method is the same, as we shall demonstrate, beginning with the diagonal elements, the first of which is the $l = 0$ case. The relevant wave function is given as follows:

$$\Psi_{200} = \frac{2}{\sqrt{4\pi}} \left(\frac{1}{2a_0}\right)^{\frac{3}{2}} \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}} \tag{2.64}$$

We shall make use of the following simplifications:

$$\bigtriangledown^2 \left(1 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}} = \frac{1}{{a_0}^2} \left(\frac{r - 8a_0}{4r}\right) \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}}, \tag{2.65}$$

$$\frac{\partial}{\partial x^i} \left(1 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}} = \frac{x^i}{a_0^2} \left(\frac{4a_0 - r}{2r(r - 2a_0)}\right) \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}}. \tag{2.66}$$

Using these results, it follows that,

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}} |\Psi> = \left[x^i \left(1 - \frac{a^2}{{a_0}^2} \left(\frac{r - 8a_0}{4r} + \frac{4a_0 - r}{2r(r - 2a_0)}\right)\right)\right]^{-\frac{1}{2}} |\Psi> . \tag{2.67}$$

The energy shift is given by:

$$e^2 < \Psi_{200} | \frac{1}{r_{new}} | \Psi_{200} >= \frac{4e^2}{8a_0^3} \int_0^\infty r^2 (1 - \frac{r}{2}) dr e^{-r} \frac{1}{r} \left[1 - \frac{a^2}{{a_0}^2} \left(\frac{r^2 - 12r + 24}{4r(r - 2)}\right)\right]^{-1} . \tag{2.68}$$

where I have already made the change to a dimensionless r. Simplifying and making the same expansion as before, we can find the shift in the energy:

$$\Delta E'_{200} = -\frac{e^2 \beta}{2a_0} \int_0^\infty r(1 - \frac{r}{2})^2 dr e^{-r} \left(\frac{r^2 - 12r + 24}{4r(r - 2)}\right). \tag{2.69}$$

$$\Delta E'_{200} = \frac{11 e^2 a^2}{16 a_0^3} \tag{2.70}$$

The prime indicates that this energy does not yet take into account the affects of the degeneracy. The $l = 1$ states are more involved because of their θ and ϕ dependence. Their wave functions are given as follows:

$$\Psi_{210} = \sqrt{\frac{1}{4\pi}} \cos\theta \left(\frac{1}{2a_0}\right)^{\frac{3}{2}} \frac{r}{a_0} e^{-\frac{r}{2a_0}} \tag{2.71}$$

$$\Psi_{211} = -\sqrt{\frac{1}{8\pi}} e^{i\phi} \sin\theta \left(\frac{1}{2a_0}\right)^{\frac{3}{2}} \frac{r}{a_0} e^{-\frac{r}{2a_0}} \tag{2.72}$$

$$\Psi_{21-1} = -\Psi^*_{211} = \sqrt{\frac{1}{8\pi}} e^{-i\phi} \sin\theta \left(\frac{1}{2a_0}\right)^{\frac{3}{2}} \frac{r}{a_0} e^{-\frac{r}{2a_0}} \tag{2.73}$$

The procedure for calculating the shifted energies is the same as before, but is complicated by the angular dependence. As these three wave functions are very similiar, it shall be convenient to express them as follows:

$$\Psi = kR(r)\Theta(\theta)\Phi(\phi), \tag{2.74}$$

where k is a constant depending on the normalization and

$$R(r) = re^{-\frac{r}{2a_0}} \tag{2.75}$$

$$\Theta_{m=0}(\theta) = \cos\theta \tag{2.76}$$

$$\Theta_{m=1}(\theta) = \sin\theta \tag{2.77}$$

$$\Phi_{m=1}(\phi) = \Phi^*_{m=-1}(\phi) = e^{i\phi} \tag{2.78}$$

The Lapacian in spherical coordinates is given as follows:

$$\bigtriangledown^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta^2}\frac{\partial^2}{\partial\phi^2} \tag{2.79}$$

We can break this up component-wise and greatly simplify our lives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r) = \left(\frac{2}{r^2} - \frac{2}{a_0 r} + \frac{1}{4{a_0}^2}\right)R(r) \tag{2.80}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta_{m=0}(\theta) = \left(-\frac{2}{r^2}\right)\Theta_{m=0}(\theta) \tag{2.81}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta_{m=1}(\theta) = \frac{1}{r^2}\left(\frac{cos^2\theta - sin^2\theta}{sin^2\theta}\right)\Theta_{m=1}(\theta) \tag{2.82}$$

$$\frac{1}{r^2\sin\theta^2}\frac{\partial^2}{\partial\phi^2}\Phi_{m=1}(\phi) = -\frac{1}{r^2}\frac{1}{sin^2\theta}\Phi_{m=1}(\phi) \tag{2.83}$$

$$\frac{1}{r^2\sin\theta^2}\frac{\partial^2}{\partial\phi^2}\Phi_{m=-1}(\phi) = -\frac{1}{r^2}\frac{1}{sin^2\theta}\Phi_{m=-1}(\phi) \tag{2.84}$$

We must additionally calculate the second term in the hamiltonian

$$\frac{\partial}{\partial x^i}R(r) = \frac{x^i}{r}\left(\frac{1}{r} - \frac{1}{2a_0}\right)R(r). \tag{2.85}$$

$$\frac{\partial}{\partial x^i}\Theta_{m=0}(\theta) = \frac{\partial}{\partial x^i}\frac{z}{r} = \left(\delta^i{}_z r - \frac{x^i}{r^2}\right)\Theta_{m=0}(\theta) \tag{2.86}$$

$$\frac{\partial}{\partial x^i}\Theta_{m=1}(\theta) = \frac{\partial}{\partial x^i}\frac{\sqrt{x^2+y^2}}{r} = \left(-\frac{x^i}{r^2} + \frac{x^i(\delta^i{}_x - \delta^i{}_y)}{x^2+y^2}\right)\Theta_{m=1}(\theta) \tag{2.87}$$

$$\frac{\partial}{\partial x^i}\Phi_{m=1}(\phi) = \frac{\partial}{\partial x^i}e^{i\tan^{-1}\frac{y}{x}} = i\frac{x^i(\delta^i{}_x - \delta^i{}_y)}{x^2+y^2}\Phi_{m=1}(\phi). \tag{2.88}$$

$$\frac{\partial}{\partial x^i}\Phi_{m=-1}(\phi) = \frac{\partial}{\partial x^i}e^{-i\tan^{-1}\frac{y}{x}} = -i\frac{x^i(\delta^i{}_x - \delta^i{}_y)}{x^2+y^2}\Phi_{m=-1}(\phi). \tag{2.89}$$

where, the Kroenecker delta function, $\delta^i{}_j, = 1$ for $i = j$ and $= 0$ for $i \neq j$. From this, it follows that,

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi_{210}> = \left[x^i - a^2\left(\frac{x^i}{4{a_0}^2} - \frac{5x^i}{2a_0 r} + \frac{\delta^i{}_z}{z}\right)\right]^{-\frac{1}{2}}|\Psi_{210}\rangle. \tag{2.90}$$

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi_{211}> = \left[x^i - a^2\left(\frac{x^i}{4{a_0}^2} - \frac{5x^i}{2a_0 r} + x^i\frac{(1+i)\delta^i{}_x + (1-i)\delta^i{}_y}{x^2+y^2}\right)\right]^{-\frac{1}{2}}|\Psi_{211}\rangle. \tag{2.91}$$

$$\frac{1}{\sqrt{(x^i + a^2 p^j x^i p^j)}}|\Psi_{211}> = \left[x^i - a^2\left(\frac{x^i}{4{a_0}^2} - \frac{5x^i}{2a_0 r} + x^i\frac{(1-i)\delta^i{}_x + (1+i)\delta^i{}_y}{x^2+y^2}\right)\right]^{-\frac{1}{2}}|\Psi_{211}\rangle. \tag{2.92}$$

We shall now proceed by calculating the energy shift for the m=0.

$$e^2 < \Psi_{210}|\frac{1}{r_{new}}|\Psi_{210} >$$

$$= e^2 < \Psi_{210}|\left[\left(x^i - a^2\left(\frac{x^i}{4{a_0}^2} - \frac{5x^i}{2a_0 r} + \frac{\delta^i{}_z}{z}\right)\right)^2\right]^{-\frac{1}{2}}|\Psi_{210} > \tag{2.93}$$

$$= e^2 < \Psi_{210}| \left[r^2 - 2a^2 \left(\frac{r^2}{4{a_0}^2} - \frac{5r}{2a_0 r} + 1 \right) + \mathcal{O}(a^4) \right]^{-\frac{1}{2}} |\Psi_{210}> \tag{2.94}$$

$$= e^2 < \Psi_{210}| \frac{1}{a_0 r} \left[1 - 2\beta \left(\frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2} \right) + \mathcal{O}(a^4) \right]^{-\frac{1}{2}} |\Psi_{210}> \tag{2.95}$$

At this point, we binomially expand and ignore the terms $\mathcal{O}(a^4)$,

$$= -\frac{e^2}{4\pi \cdot 8a_0} \int_0^\infty r^3 e^{-r} cos^2\theta \sin\theta \left[1 + \beta \left(\frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2} \right) \right], \tag{2.96}$$

where I've changed to a dimensionless r. After a little simplification, we can find the shifted energy

$$\Delta E'_{210} = \frac{5e^2\beta}{48a_0} \tag{2.97}$$

The energy shift for the m = 1 case follows simliarly,

$$e^2 < \Psi_{211}| \frac{1}{r_{new}} |\Psi_{211}> =$$

$$= e^2 < \Psi_{211}| \left[\left(x^i - a^2 \left(\frac{x^i}{4{a_0}^2} - \frac{5x^i}{2a_0 r} + x^i \frac{(1+i)\delta^i{}_x + (1-i)\delta^i{}_y}{x^2+y^2} \right) \right)^2 \right]^{-\frac{1}{2}} |\Psi_{211}> \tag{2.98}$$

$$= e^2 < \Psi_{211}| \left[r^2 - 2a^2 \left(\frac{r^2}{4{a_0}^2} - \frac{5r}{2a_0} + \frac{x^2+y^2+i(x^2-y^2)}{x^2+y^2} + \mathcal{O}(a^4) \right)^2 \right]^{-\frac{1}{2}} |\Psi_{211}> \tag{2.99}$$

After some simplifications, we can find the shifted energy

$$\Delta E'_{210} = -\frac{e^2\beta}{64\pi a_0} \int_0^\infty r^3 e^{-r} \sin^3\theta \left(\frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2} + i\frac{(\cos\phi + \sin\phi)}{r^2} \right), \tag{2.100}$$

which leads to:

$$\Delta E'_{211} = \frac{5e^2\beta}{48a_0} \tag{2.101}$$

It's easy enough to see, refering to Eq. (2.101), that

$$\Delta E'_{21-1} = \Delta E'_{211} = \frac{5e^2\beta}{48a_0}. \tag{2.102}$$

This completes our study of the diagonal elements of our shifted energy matrix. We now have to consider the off-diagonal elements. We'll begin by considering the

first row, which corresponds to combinations of the $|\Psi_{200}>$ state with $l = 1$ states. In normal quantum mechanics, we would expect these elements to be zero because the $|\Psi_{200}>$ state is angularly independent; however, since it is uncertain what role exactly our modification will play, we explicitly compute the shifts. Once again, I've switched to a dimensionless r,

$$e^2 < \Psi_{200}|\frac{1}{r_{new}}|\Psi_{210}>=$$

$$< \Psi_{200}|\left[r^2 - \beta r^2\left(\frac{r^2-12r+24}{4r(r-2)} + \frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2}\right) + \mathcal{O}(\beta^2)\right]^{-\frac{1}{2}}|\Psi_{210}> \tag{2.103}$$

After binomial expansion, we find a shifted energy

$$\frac{1}{64\pi}\int \sin\theta\cos\theta\left(\frac{r^2-12r+24}{4r(r-2)} + \frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2}\right)d\theta d\phi dr \tag{2.104}$$

Because the theta integral trivally vanishes, we're left with no energy shift,

$$\Delta E'_{200,210} = 0. \tag{2.105}$$

It can easily be shown that

$$\Delta E'_{200,210} = \Delta E'_{210,200}. \tag{2.106}$$

We shall now investigate the m = 1 case:

$$e^2 < \Psi_{200}|\frac{1}{r_{new}}|\Psi_{211}>=$$

$$< \Psi_{200}|\left[r^2 - \beta r^2\left(\frac{r^2-12r+24}{4r(r-2)} + \frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2} + i\frac{\cos\phi - \sin\phi}{r^2}\right) + \mathcal{O}(\beta^2)\right]^{-\frac{1}{2}}|\Psi_{211}> \tag{2.107}$$

The shifted energy, $\Delta E'_{200,211} =$

$$\frac{\sqrt{2}\beta}{64\pi a_0}\int \sin^2\theta e^{i\phi}r^2\left(1-\frac{r}{2}\right)\left(\frac{r^2-12r+24}{4r(r-2)} + \frac{1}{4} - \frac{5}{2r} + \frac{1}{r^2} + i\frac{\cos\phi - \sin\phi}{r^2}\right)d\theta d\phi dr \tag{2.108}$$

We inspect the integral and immediately eliminate all terms in parenthesis without ϕ dependence. Evalulating the remaining term yields

$$\Delta E'_{200,211} = \frac{\sqrt{2}(1+i)e^2\beta\pi}{256a_0}. \tag{2.109}$$

We can again argue:

$$\Delta E'_{211,200} = \Delta E'^*_{200,211} = \frac{\sqrt{2}(1-i)e^2\beta\pi}{256a_0}. \tag{2.110}$$

Furthermore, because of the relation between the $m = 1$ case and the $m = -1$ case,

$$\Delta E'_{200,21-1} = -\Delta E'^*_{200,211} = -\frac{\sqrt{2}(1-i)e^2\beta\pi}{256a_0}. \tag{2.111}$$

$$\Delta E'_{21-1,200} = \Delta E'^*_{200,21-1} = -\frac{\sqrt{2}(1+i)e^2\beta\pi}{256a_0}. \tag{2.112}$$

We've now effectively completed the first row and column of our matrix, as well as the diagonal elements. This leaves 6 terms to consider – wave functions with l = 1 but different values of m.

$$e^2 < \Psi_{210}|\frac{1}{r_{new}}|\Psi_{211} >=$$

$$< \Psi_{210}| \left[r^2 - \beta r^2 \left(\frac{1}{2} - \frac{5}{r} + 2\frac{1}{r^2} + i\frac{\cos\phi - \sin\phi}{r^2} \right) + \mathcal{O}(\beta^2) \right]^{-\frac{1}{2}} |\Psi_{211} > \tag{2.113}$$

$$< \Psi_{210}|\frac{1}{r} \left[1 - \beta \left(\frac{1}{2} - \frac{5}{r} + \frac{2}{r^2} + i\frac{\cos\phi - \sin\phi}{r^2} \right) + \mathcal{O}(\beta^2) \right]^{-\frac{1}{2}} |\Psi_{211} > \tag{2.114}$$

The shifted energy, $\Delta E'_{210,211} =$

$$\frac{\beta}{64\sqrt{2}\pi a_0} \int r^3 e^{-r} e^{i\phi} \sin^2\theta \cos\theta \left[\frac{1}{2} - \frac{5}{r} + \frac{2}{r^2} + i\frac{\cos\phi - \sin\phi}{r^2} \right] d\theta d\phi dr. \tag{2.115}$$

Because of the theta integral, this trivially vanishes

$$\Delta E'_{210,211} = E'_{211,210} = 0 \tag{2.116}$$

Similarly, because of the relationship between Ψ_{211} and Ψ_{21-1},

$$\Delta E'_{210,21-1} = \Delta E'_{21-1,210} = 0 \tag{2.117}$$

This leaves us with essentially one more matrix element to compute:

$$e^2 < \Psi_{21-1}|\frac{1}{r_{new}}|\Psi_{211} >=$$

$$= e^2 < \Psi_{21-1}| \left[r^2 - \beta \left(\frac{r^2}{2{a_0}^2} - \frac{5r}{a_0} + \frac{2(x^2+y^2) + i(x^2-y^2) - i(x^2-y^2)}{x^2+y^2} \right) \right]^{-\frac{1}{2}} |\Psi_{211} > \tag{2.118}$$

where I've chosen not to write $\mathcal{O}(\beta^2)$ terms.

$$= e^2 < \Psi_{21-1}|\frac{1}{r} \left[1 - a^2 \left(\frac{1}{2{a_0}^2} - \frac{5}{a_0 r} + \frac{2}{r} \right) \right]^{-\frac{1}{2}} |\Psi_{211} > \tag{2.119}$$

We note that $< \Psi_{21-1}|\Psi_{211} >= 0$ because of the ϕ integral. Since there is no ϕ dependence in our integral Eq. (2.119),

$$\Delta E'_{21-1,211} = 0. \tag{2.120}$$

Similarly,

$$\Delta E'_{21-1,211} = \Delta E'_{211,21-1} = 0. \tag{2.121}$$

As with the $n = 2$ case of the harmonic oscillator, we have to account for the degeneracy. The shifted energies can be found by diagonalizing the following 4x4 matrix:

$$W_2 = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{bmatrix} \tag{2.122}$$

The basis of the matrix W_2 is $|\Psi_{200} >$, $|\Psi_{21-1} >$, $|\Psi_{210} >$, $|\Psi_{211} >$. It can be shown that:

$$W_2 = e^2 \frac{a^2}{{a_0}^3} \begin{bmatrix} \frac{11}{16} & -\frac{\sqrt{2}(1-i)\pi}{256} & 0 & \frac{\sqrt{2}(1+i)\pi}{256} \\ -\frac{\sqrt{2}(1+i)\pi}{256} & \frac{5}{48} & 0 & 0 \\ 0 & 0 & \frac{5}{48} & 0 \\ \frac{\sqrt{2}(1-i)\pi}{256} & 0 & 0 & \frac{5}{48} \end{bmatrix} \tag{2.123}$$

Upon diagonalization, we find the following shifted energies:

$$\Delta E_{200} = \left(19 + \frac{\sqrt{12544 + 18\pi^2}}{8}\right) \frac{e^2 a^2}{48 a_0^3} \tag{2.124}$$

$$\Delta E_{211} = \Delta E_{21-1} = \left(19 - \frac{\sqrt{12544 + 18\pi^2}}{8}\right) \frac{e^2 a^2}{48 a_0^3} \tag{2.125}$$

$$\Delta E_{210} = \frac{5 e^2 a^2}{48 a_0^3}, \tag{2.126}$$

which is not in agreement with others [9]. I am currently trying to resolve this discrepancy.

Chapter 3

Physical Models and Placing Bounds on the Small Parameter

3.1 Modeling the 1-d Harmonic Oscillator with a Penning Trap

Our result for the one dimensional harmonic oscillator gave an energy shift Eq. (2.18), which, for large n depends on n^2, and clearly departs from normal quantum mechanics. We can therefore express our large n energy shift as follows

$$\frac{\Delta E_n}{\omega} = \frac{1}{2} m\omega a^2 n^2. \tag{3.1}$$

A physical example of a one dimensional harmonic oscillator is the cyclotron motion of an electron in a Penning trap. This system has been explored by others [8], [9], and we shall reproduce the results. The cyclotron frequency of an electron in a magnetic field is given by setting the Lorentz force equal to the centripetal force

$$evB = mv\omega. \tag{3.2}$$

It follows that the cyclotron frequency is

$$\omega_c = \frac{eB}{m}. \tag{3.3}$$

If we assume that it is physically possible to measure shifts in the energy on the order of ω_c, then we can set the following bound using Eq. (3.1)

$$1 > \frac{1}{2} eBa^2 n^2 \tag{3.4}$$

$$a < \sqrt{\frac{2}{eB}}\frac{1}{n}. \tag{3.5}$$

In order for the electron to remain non-relativisitic, we must require

$$\frac{n\omega}{mc^2} << 1. \tag{3.6}$$

Thus, a good approximation for an acceptable maximum value of n is

$$n = \frac{(mc)^2}{10eB} \tag{3.7}$$

In a magnetic field of 6T, we can combine the results of Eq. (3.5) and Eq. (3.7) and attain a bound on a

$$a < \frac{\sqrt{2eB}}{(mc)^2} \tag{3.8}$$

$$a > 1 \text{ GeV}^{-1} \tag{3.9}$$

It should be stressed however, that making a precise measurement on an extremely excited electron is not trivial.

3.2 The 1S-2S Splitting of the Hydrogen Atom

When an electron drops down from the excited 2S state to the ground state, 1S, a photon whose wavelength can be measured to an extraordinary precision is emitted. We use uncertainty in experimental measurements [14] of this energy splitting to bound our parameter, a.

$$\Delta E = \Delta E_{200} - \Delta E_{100} < h\nu \tag{3.10}$$

When we evaluate Eq. (3.10) with $a_0 = 2.681 \times 10^5$ GeV^{-1}, we can place the following bound:

$$a < 1.214 \times 10^{-2} \text{ GeV}^{-1} \tag{3.11}$$

which is a much tighter bound than we were able to attain with the Penning trap.

Chapter 4

Conclusions

We were able to modify the position operator which yielded a quantum theory with a minimal length uncertainty. Using perturbation theory, we efficiently calculated shifted energy eigenvalues for the one and three dimensional harmonic oscillator, as well as the ground and first excited state of the hydrogen atom. These shifted energy eigenvalues were found to be in agreement with the literature for the harmonic oscillator [7-8], obtained using much more complication methods. Although we have a discrepancy with [9] in the case of the hydrogen atom, it should be noted that our theory is profoundly simpler [1].

Possible future work includes understanding the descrepancy as well as perhaps modeling corrections to energy levels of diatomic molecules assuming a morse potential.

[1] We recently became aware that another group is also in disagreement with [9]

Bibliography

[1] E. Witten: *Reflections on the Fate of Spacetime* Physics Today, April 1996, page 24;

[2] A. Connes: *Publ IHES* **62** (1986) 257;

[3] A. Connes: *Geometric noncommutative* Inter Edititions, Paris, 1990;

[4] M. Demetrian, D. Kochan: *Quantum Mechanics on non-commutative plane*, hep-th/0102050;

[5] M. Maggiore, Phys. Lett. B **319**, 89 (1993); Phys. Rev. D **49**, 5182 (1994);

[6] S. Gasiorwicz: *Quantum Physics*, 3rd Ed. John Wiley & Sons, Inc 2003;

[7] A. Kempf, G. Mangano, R. Mann: *Hilbert space representation of minimal length uncertainty relation*, Phys.Rev.D **52**, 1108 (1995);

[8] L. Chang, D. Minic, N. Okamura, T. Takeuchi: *Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations*, Phys.Rev.D **65**, 125027 (2002);

[9] R. Akhoury and Y.P Yao, *Minimal Length Uncertainty Relation and the Hydrogen Spectrum*, hep-ph/0302108 v2 20 Feb 2003;

[10] Snyder *Quantized Space-Time*, Phys. Rev 71 (1947) 38.

[11] L.S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 223 (1986);

[12] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products, Sixth Edition* (Academic Press, San Diego, California, 2000);

[13] Brau, *Minimal Length Uncertainty Relation and the Hydrogen Atom*, arXiv:quant-ph/9905033 v1 10 May 1999;

[14] M. Niering et. al., Phys Rev. Lett. **84** 2000 5496;